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LETTER TO THE EDITOR

A standard form for generalised CP transformations†

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Abstract. The investigation of general CP transformations leads to transformations of the form $U \to W^{\mathrm{T}} U W$ with unitary matrices U, W. It is shown that a basis of weak eigenstates can always be chosen such that $W^{\mathrm{T}} U W$ has a certain real standard form.

To study CP violation in theories with several fermion families it is necessary to consider general CP transformations. In the standard model (Glashow 1961, Weinberg 1967, Salam 1968) and its extensions like the left-right symmetric model (Pati and Salam 1975, Mohapatra and Pati 1975, Mohapatra and Senjanović 1980) such a transformation acting on the quark or lepton fields has the form (Ecker *et al* 1981, 1984, 1987)

$$\psi(x) \to U_{\mathrm{CP}} C \psi^*(x^0, -\boldsymbol{x}) \tag{1}$$

where the vector of n_{G} fermion fields ψ (n_{G} is the number of generations) has a definite chirality, U_{CP} is a unitary $n_{\mathrm{G}} \times n_{\mathrm{G}}$ matrix acting in generation space and C is the Dirac charge-conjugation matrix. The CP transformation of the Higgs fields has a similar form. General CP transformations cannot only be used to constrain Yukawa couplings but they are also necessary for investigating if a given Lagrangian is CP invariant or not (Bernabéu *et al* 1986). In both cases, basis transformations of the weak eigenstates are extremely useful. Redefining the fermion fields by

$$\psi = W_\psi \psi' \tag{2a}$$

the CP matrix U_{CP} in the new basis is given by‡

$$U'_{\mathrm{CP}} = W_\psi^\dagger U_{\mathrm{CP}} W_\psi^* \tag{2b}$$

with W_ψ being a unitary matrix. Since basis transformations cannot change the physical content of a model we can question whether U_{CP} can be brought to a certain simple 'standard' form by applying the transformation (2*b*). The following theorem shows that this is indeed the case.

Theorem. To every unitary $n \times n$ matrix U there exists a unitary matrix W such that

$$W^{\mathrm{T}} U W = \begin{pmatrix} 0_1 & & & \\ & \ddots & & \\ & & 0_l & \\ & & & \mathbf{1}_m \end{pmatrix} \qquad 2l + m = n \tag{3}$$

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‡ In the case of time reversal, one obtains the same structure of basis transformations.

with 2×2 orthogonal matrices

$$0_\nu = \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix} \qquad 0<\theta_\nu \leqslant \pi/2;\ \nu=1,\ldots,l \tag{4}$$

and with the m-dimensional unit matrix $\mathbf{1}_m$. Furthermore, the angles θ_ν are determined by the Hermitian matrix

$$\tfrac{1}{4}(U+U^{\mathrm{T}})^{\dagger}(U+U^{\mathrm{T}}) \tag{5}$$

which has twice degenerate eigenvalues $\cos^2\theta_\nu$ ($\nu=1,\ldots,l$) and an m-fold degenerate eigenvalue 1.

Before proving the theorem we want to make a few remarks. For complex matrices U one cannot apply the theorems on normal matrices since transformations of the type (2b) are not the usual similarity transformations. However, if U is real one can use the well known properties of real normal matrices to find an orthogonal W which brings U to the desired form (3). Therefore, the essential result of the theorem is that a general complex unitary U can always be made real by a transformation of the type (2b). Applying the theorem to a unitary 3×3 matrix illustrates nicely its main features: for every U there exists a unitary matrix W such that

$$W^{\mathrm{T}}UW = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad 0\leqslant\theta\leqslant\pi/2.$$

Furthermore, the matrix (5) has the eigenvalues $\cos^2\theta$ (twofold degenerate) and 1.

In order to prove the theorem the matrix U can be decomposed into a sum of a symmetric and an antisymmetric matrix by

$$U=\tfrac{1}{2}(U+U^{\mathrm{T}})+\tfrac{1}{2}(U-U^{\mathrm{T}}). \tag{6}$$

Then there exists a unitary matrix W_1 such that (Schur 1945)

$$W_1^{\mathrm{T}}\tfrac{1}{2}(U+U^{\mathrm{T}})W_1 = D \tag{7}$$

with D being diagonal and positive. Furthermore, one can write

$$W_1^{\mathrm{T}}\tfrac{1}{2}(U-U^{\mathrm{T}})W_1 = A+\mathrm{i}B \tag{8}$$

with real and antisymmetric matrices A, B. Exploiting the unitarity of U leads to the four real equations

$$[D,A]=0 \tag{9a}$$

$$\{D,B\}=0 \tag{9b}$$

$$[A,B]=0 \tag{9c}$$

$$D^2-A^2-B^2=\mathbf{1} \tag{9d}$$

which are the starting point for further discussion.

By an appropriate permutation of the basis vectors of $\mathbb{C}^n$, D can always be arranged to have the form

$$D=\begin{pmatrix} d_0\mathbf{1}_{n_0} & & & \\ & d_1\mathbf{1}_{n_1} & & \\ & & \ddots & \\ & & & d_k\mathbf{1}_{n_k} \end{pmatrix} \tag{10}$$

with $0=d_0<d_1<\ldots<d_k$ and $n_0+n_1+\ldots+n_k=n$. n_0 may be zero but $n_i>0$ for $i=1,\ldots,k$. Now it follows from $(9a)$ that A decays into blocks of $n_i\times n_i$ matrices A_i $(i=0,\ldots,k)$:

$$A=\begin{pmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}. \tag{11}$$

From $(9b)$ we obtain

$$B=\begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{12}$$

with an $n_0\times n_0$ matrix B_0. Of cours., A_i $(i=0,\ldots,k)$ and B_0 are all antisymmetric matrices.

$W_1^{\mathrm{T}}UW_1$ consists of the blocks $A_0+\mathrm{i}B_0$ and $d_i\mathbf{1}_{n_i}+A_i$ $(i=1,\ldots,k)$ which can now be discussed separately. Since $-A_i^2$ is positive $(9d)$ implies $d_k\leqslant 1$. If $d_k=1$ then necessarily $A_k=0$, $n_k\equiv m$ and A_i is non-singular for $i=1,\ldots,k-1$. For $d_k<1$ also A_k is non-singular and $m=0$. Let us now assume $0<d_i<1$. By an orthogonal matrix we can transform A_i to†

$$\begin{pmatrix} 0 & \lambda_i^{(1)} & & & \\ -\lambda_i^{(1)} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_i^{(n_i/2)} \\ & & & -\lambda_i^{(n_i/2)} & 0 \end{pmatrix} \tag{13}$$

with $\lambda_i^{(\alpha)}>0$ $(\alpha=1,\ldots,n_i/2)$. As a consequence of the orthogonality relation $(9d)$ we obtain

$$\lambda_i^{(1)}=\ldots=\lambda_i^{(n_i/2)}=(1-d_i^2)^{1/2}. \tag{14}$$

Thus we have arrived at the desired form (3) for the blocks with $i=1,\ldots,k$. In each of these blocks the θ_ν are all equal with $d_i=\cos\theta_\nu$.

It remains to discuss the matrix $A_0+\mathrm{i}B_0$. Since A_0, B_0 commute $(9c)$ one can show that both can be brought to the form (13) by a common orthogonal transformation. For the non-zero matrix elements in (13) we write $\lambda_0^{(\alpha)}$ and $\mu_0^{(\alpha)}$ $(\alpha=1,\ldots,n_0/2)$ for A_0 and B_0, respectively. Since $A_0+\mathrm{i}B_0$ is unitary we have

$$|\lambda_0^{(\alpha)}+\mathrm{i}\mu_0^{(\alpha)}|=1 \tag{15}$$

and thus by a diagonal phase matrix $\tilde{W}$ we can achieve

$$\tilde{W}(A_0+\mathrm{i}B_0)\tilde{W}=\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}. \tag{16}$$

In this way also the zeroth block has been brought to the form (3) with all angles being $\pi/2$ in this block. Finally, from (7) it is clear that the matrix (5) determines the angles of the orthogonal matrices (4). This completes the proof of the theorem. The transformation W is obtained as the product of all the transformations performed at each step of the proof.

† n_i must be even because A_i is antisymmetric and non-singular.

Note added. In general, the most straightforward way to determine the parameters θ_ν of the standard form (3) is to diagonalise the matrix UU^* with eigenvalues $\exp(\pm 2i\theta_\nu)$ ($\nu = 1, \ldots, l$) and 1 (m-fold degenerate).

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